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The Global Dimension of Boolean Rings*

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1. INTRODUCTION

It is well known that regular rings are characterized by the property of having weak global dimension zero (see [*2*] or [*3*]). In this paper we study the ordinary global dimension of regular rings, and, in particular, Boolean rings. It is an easy consequence of some standard results of homological algebra that if R is a regular ring of cardinality $\aleph_n$, then gl. dim $R \leqslant n + 1$. Our main result is that this estimate is sharp. To be precise, the free Boolean algebra on $\aleph_n$ generators has global dimension $n + 1$.

All rings are assumed to be associative with an identity element. By a module, we mean a left module, so that when speaking of the dimension of M (abbreviated h. dim M), we mean its left homological dimension. Our standard reference on homological algebra is [*6*].

A regular ring R is one which is regular in Von Neumann's sense: for each $a \in R$, there exists $b \in R$ such that $aba = a$. A Boolean ring is a ring which satisfies the identity $x^2 = x$. Every Boolean ring is commutative and regular.

We will consider ordinal numbers as sets, namely the set of all smaller ordinal numbers. Cardinal numbers are particular ordinal numbers. The cardinality of a set S is denoted by $|S|$.

If M is an R-module, and $X = \{x_i \mid i \in I\} \subseteq M$, we denote by $[X]$ or $[\{x_i \mid i \in I\}]$ the submodule of M generated by X. Thus, $[X]$ consists of all finite sums

$$r_0 x_{i_0} + \cdots + r_{n-1} x_{i_{n-1}},$$

where the r_j are elements of R and $i_j \in I$. We use $M \oplus N$ to denote the direct sum of the modules M and N. More generally, $\oplus \Sigma_{i \in I} M_i$ stands for the direct sum of the set $\{M_i \mid i \in I\}$.

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2. The Upper Estimate

We will use a result of Auslander to obtain an upper bound for the global dimension of all regular rings of cardinality not exceeding $\aleph_n$. The result is doubtless well known, but it does not seem to be in the literature.

Lemma 2.1. *Let R be a ring with the property that every countably generated left ideal of R has projective dimension $\leqslant m$. If J is a left ideal of R which is generated by a set of cardinality $\aleph_n$, then*

$$\text{h. dim } J \leqslant m + n.$$

Proof. If $n = 0$, then the conclusion of the lemma coincides with the hypothesis. We may therefore assume that $n > 0$, and make the induction hypothesis that the lemma is true for ideals generated by sets of cardinality less that $\aleph_n$. Let $\{x_\xi \mid \xi < \lambda\}$ be a set of generators of J indexed by the cardinal number λ. For $\eta < \lambda$, define $B_\eta = [\{x_\xi \mid \xi < \eta\}]$. Then B_η is a left ideal of R which satisfies the induction hypothesis, so that h. dim $B_\eta \leqslant m + n - 1$. Consequently, h. dim $B_{\eta+1}/B_\eta \leqslant m + n$. We now note that the well ordered sequence $\{B_\xi \mid \xi < \lambda\}$ of submodules of J satisfy the conditions of Auslander's Proposition 3 (in [*1*]). Thus, h. dim $J \leqslant m + n$.

Corollary 2.2. *Let R be a regular ring of cardinality no greater than $\aleph_n$. Then*

$$\text{gl. dim } R \leqslant n + 1.$$

Proof. Kaplansky has shown (see [*4*]) that every countably generated left ideal in a regular ring is projective. Thus h. dim $J \leqslant n$ for every left ideal J in R by the lemma. This implies that gl. dim $R \leqslant n + 1$ (see [*1*]).

3. Basis Theorem

If R is a regular ring, then every projective R-module is a direct sum of principal left ideals of R (see [*5*]). We will use this result to prove the following theorem which is needed to show that the upper estimate for the global dimension given in Section 2 is sharp.

Theorem 3.1. *Let R be a regular ring, and suppose that P is a projective R-module. Assume that P is generated by the set $Y = \{y_i \mid i \in I\}$. Then there is a family $\{e_i \mid i \in I\}$ of idempotents in R such that*

$$P = \oplus \sum_{i \in I} Re_i y_i .$$

The proof of 3.1 uses the following result. We preserve the notation and hypotheses of 3.1.

LEMMA 3.2. *Let S be a direct summand of the projective R-module T, and suppose that $y \in T$. Then $Ry + S$ is a direct summand of T, and there is an idempotent $e \in R$ such that $Ry + S = Rey \oplus S$.*

Proof. Let $T = Q \oplus S$. Write $y = q + s$, where $q \in Q$ and $s \in S$. Then Rq is a direct summand of Q (see [5], Lemma 4), say $Q = W \oplus Rq$. Hence $T = W \oplus (Rq + S) = W \oplus (Ry + S)$. Moreover, Rq is isomorphic to a principal left ideal of R, so that there is an idempotent $e \in R$ such that $rq = 0$ if and only if $re = 0$. In particular, $q = eq$, and therefore

$$Ry + S = Req + S = Rey + S.$$

Moreover, $rey \in S$ implies $req \in S \cap Q = 0$. Thus, $req = 0$, and consequently $re = 0$. This shows that $Ry + S = Rey \oplus S$.

By a straightforward induction we obtain the following consequence of 3.2.

COROLLARY 3.3. *Let S be a direct summand of the projective R-module T such that T/S is countably generated. Suppose that $\{y_j \mid j \in J\} \subseteq T$ is such that $S \cup \{y_j \mid j \in J\}$ generates T. Then there exist idempotents $e_j \in R$ such that*

$$T = \Big(\oplus \sum_{j \in J} Re_j y_j\Big) \oplus S.$$

It is now possible to give the proof of Theorem 3.1. The technique is similar to the method used by Kaplansky in [5] (see the proof of Theorem 1 of that paper).

By Theorem 4 of [5], there is a set $\{x_k \mid k \in K\} \subseteq P$ such that $P = \oplus \sum_{k \in K} Rx_k$. Let $\pi_k : P \to Px_k$ be the associated projection homomorphism. For $L \subseteq K$, denote $S(L) = [\{x_k \mid k \in L\}]$. Let $L \subseteq K$ be countable. For each $k \in L$, select a finite set $I_k \subseteq I$ so that $x_k \in [\{y_i \mid i \in I_k\}]$. Let $L' = L \cup \bigcup_{k \in K} \bigcup_{i \in I_k} \{k' \in K \mid \pi_{k'}(y_i) \neq 0\}$. Then L' is countable, and $S(L) \subseteq [Y \cap S(L')]$. If this process is iterated, we obtain a nested sequence $L_0 = L$, $L_1 = L_0'$, $L_2 = L_1'$,... of countable subsets of K such that $S(L_n) \subseteq [Y \cap S(L_{n+1})]$. The union $\bar{L}$ of this sequence is countable and satisfies $S(\bar{L}) = [Y \cap S(\bar{L})]$. Using this construction, we can build a transfinite sequence $\{L_\xi \mid \xi < \lambda\}$ of subsets of K such that $L_\xi \subseteq L_\eta$ for $\xi \leqslant \eta$, $L_\eta = \bigcup_{\xi < \eta} L_\xi$ if η is a limit ordinal, $\bigcup_{\xi < \lambda} L_\xi = I$, $|L_{\xi+1} - L_\xi| \leqslant \aleph_0$ for all $\xi < \lambda$, and $S(L_\xi) = [Y \cap S(L_\xi)]$ for all $\xi < \lambda$. By transfinite induction, it is easy to show that $S(L_\eta). = \oplus \sum_{\xi < \eta} S(L_{\xi+1} - L_\xi)$ for all η. In particular, $P = \oplus \sum_{\xi < \lambda} S(L_{\xi+1} - L_\xi)$ Note that $S(L_{\xi+1})/S(L_\xi) \cong S(L_{\xi+1} - L_\xi)$ is countably generated. Moreover, there is a set $Y_\xi \subseteq Y \cap S(L_{\xi+1}) - Y \cap S(L_\xi)$

such that $S(L_\xi) \cup Y_\xi$ generates $S(L_{\xi+1})$. Hence, by Corollary 3.3, there exist idempotents $e_i \in R$ corresponding to the $y_i \in Y_\xi$ so that

$$S(L_{\xi+1}) = \left(\oplus \sum Re_iy_i\right) \oplus S(L_\xi),$$

where the sum is over Y_ξ. Carrying out this process for all $\xi < \lambda$, we obtain $P = \oplus \sum_{i \in I} Re_iy_i$.

4. A Projective Resolution

We now specialize our investigation to the case in which R is a Boolean ring, and proceed to construct a convenient projective resolution for an ideal J of R.

Let $E = \{e_\xi \mid \xi < \lambda\}$ be a set of generators of J, indexed by the ordinal number λ. For $n \geqslant 1$, let

$$\Delta_n = \{(\xi_0, \xi_1, \ldots, \xi_{n-1}) \mid \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \lambda\}.$$

Corresponding to $\delta = (\xi_0, \xi_1, \ldots, \xi_{n-1}) \in \Delta_n$, define

$$N(\delta) = e_{\xi_0} e_{\xi_1} \cdots e_{\xi_{n-1}}.$$

Let P_n denote the set of all $\varphi \in R^{\Delta_n}$ such that $\varphi(\delta) = 0$ for all except finitely many $\delta \in \Delta_n$, and $\varphi(\delta) \in RN(\delta)$ for all $\delta \in \Delta_n$. Plainly, P_n is closed under pointwise addition and multiplication by elements of R. Thus, P_n is an R-module. Moreover, P_n is projective. In fact, $P_n = \oplus \sum_{\delta \in \Delta_n} Rx_\delta$, where x_δ is defined by $x_\delta(\delta') = 0$ if $\delta' \neq \delta$, and $x_\delta(\delta) = N(\delta)$.

It is convenient to associate with each $\varphi \in P_n$ a function $\tilde{\varphi} \in R^{\lambda^n}$, defined by $\tilde{\varphi}(\xi_0, \xi_1, \ldots, \xi_{n-1}) = 0$ if $\xi_i = \xi_j$ for some $i \neq j$,

$$\tilde{\varphi}(\xi_0, \xi_1, \ldots, \xi_{n-1}) = \varphi(\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_{n-1}}),$$

if $\xi_{i_0} < \xi_{i_1} < \cdots < \xi_{i_{n-1}}$. Clearly, $\tilde{\varphi}$ is zero except on a finite set.

Define

$$\epsilon : P_1 \to J \qquad \text{by} \qquad \epsilon(\varphi) = \sum_{\xi < \lambda} \varphi(\xi).$$

If $n > 1$, define

$$d_n : P_n \to P_{n-1} \qquad \text{by} \qquad (d_n\varphi)(\delta) = \sum_{\xi < \lambda} \tilde{\varphi}(\delta, \xi)$$

for $\delta \in \Delta_{n-1}$. These mappings are evidently module homomorphisms.

LEMMA 4.1. *The sequence*

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{\epsilon} J \to 0$$

is a projective resolution of J.

Proof. Since E generates J, the mapping ϵ is surjective. For $n > 2$, $\delta \in \Delta_{n-2}$, and $\varphi \in P_n$,

$$\begin{aligned}(d_{n-1}\, d_n\varphi)(\delta) &= \sum_{\xi<\lambda}\sum_{\eta<\lambda} \tilde{\varphi}(\delta, \xi, \eta)\\ &= \sum_{\xi<\eta<\lambda} \tilde{\varphi}(\delta, \xi, \eta) + \sum_{\eta<\xi<\lambda} \tilde{\varphi}(\delta, \xi, \eta)\\ &= \sum_{\xi<\eta<\lambda} \tilde{\varphi}(\delta, \xi, \eta) + \sum_{\eta<\xi<\lambda} \tilde{\varphi}(\delta, \eta, \xi) = 0\end{aligned}$$

because of the definition of $\tilde{\varphi}$ and the fact that R has characteristic two. A similar calculation shows that $\epsilon d_2 = 0$. Let $\varphi \in P^{n-1}$ satisfy $d_{n-1}(\varphi) = 0$ (or $\epsilon(\varphi) = 0$ if $n = 2$). Let $\xi < \lambda$ be fixed for a moment. Define $\psi_\xi \in P_n$ by

$$\psi_\xi(\xi_0, \ldots, \xi_{r-1}, \xi, \xi_{r+1}, \ldots, \xi_{n-1}) = e_\xi\varphi(\xi_0, \xi_1, \ldots, \xi_{r-1}, \xi_{r+1}, \ldots, \xi_{n-1}),$$

and $\psi_\xi(\delta) = 0$ if ξ does not occur in δ. If

$$\delta' = (\xi_0, \ldots, \xi_{r-1}, \xi, \xi_{r+1}, \ldots, \xi_{n-2}) \in \Delta_{n-1},$$

then

$$\begin{aligned}(d_n\psi_\xi)(\delta') &= \sum_{\eta<\lambda} \tilde{\psi}_\xi(\delta', \eta) = \sum_{\xi\neq\eta<\lambda} e_\xi\tilde{\varphi}(\delta'', \eta)\\ &= \sum_{\eta<\lambda} e_\xi\tilde{\varphi}(\delta'', \eta) + e_\xi\varphi(\delta') = e_\xi(d_{n-1}\varphi)(\delta'') + e_\xi\varphi(\delta') = e_\xi\varphi(\delta')\end{aligned}$$

[where δ'' denotes $(\xi_0, \ldots, \xi_{r-1}, \xi_{r+1}, \ldots, \xi_{n-2})$ for convenience]. On the other hand, if ξ does not occur in δ', then $(d_n\psi_\xi)(\delta') = e_\xi\varphi(\delta')$ directly. Hence, $d_n\psi_\xi = e_\xi\varphi$. Let

$$\Phi = \{\xi_0, \xi_1, \ldots, \xi_{k-1}\}$$

be a finite subset of λ such that

$$\varphi(\delta) = 0 \quad \text{if} \quad \delta \in \Delta_{n-1} - \Phi^{n-1}.$$

Then

$$(e_{\xi_0} \vee e_{\xi_1} \vee \cdots \vee e_{\xi_k})\,\varphi(\delta) = \varphi(\delta) \qquad \text{for every} \qquad \delta \in \Delta_{n-1}.$$

Define

$$\psi = \psi_{\xi_0} + (1 - e_{\xi_0})\,\psi_{\xi_1} + \cdots + (1 - e_{\xi_0}) \cdots (1 - e_{\xi_{k-1}})\,\psi_{\xi_k} \in P_n\,.$$

We have

$$\begin{aligned} d_n\psi &= (e_{\xi_0} + (1 - e_{\xi_0})\,e_{\xi_1} + \cdots + (1 - e_{\xi_0}) \cdots (1 - e_{\xi_{k-1}})\,e_{\xi_k})\,\varphi \\ &= (e_{\xi_0} \vee e_{\xi_1} \vee \cdots \vee e_{\xi_k})\,\varphi = \varphi. \end{aligned}$$

5. The Lower Estimate

Our objective in this section is to show that there are Boolean rings of cardinality $\aleph_n$ whose global dimension is exactly $n + 1$. It is not hard to find a likely candidate for this study. Let F_n be the Boolean ring which is freely generated by the set $\{e_\xi \mid \xi < \lambda\}$ of cardinality $\aleph_n$. Then $|F_n| = \aleph_n$. If R is any Boolean ring of cardinality $\aleph_n$, and if J is an ideal of R, then there is a homomorphism of F_n to R which maps the ideal M of F_n generated by $\{e_\xi \mid \xi < \lambda\}$ onto J. Under these circumstances, Harada has shown (see [3], Proposition 7) that

$$\text{h. dim}_R\, J \leqslant \text{h. dim}_{F_n}\, M.$$

Therefore, what has to be shown is that

$$\text{h. dim}_{F_n}\, M = n.$$

The result which we will prove is somewhat more general than this.

THEOREM 5.1. *Let R be a Boolean ring, and J an ideal of R which is generated by the set $E = \{e_\xi \mid \xi < \lambda\}$, such that $|E| \geqslant \aleph_n$. Assume that*

(a) *$\{e_\xi \mid \xi < \lambda\}$ is independent, that is, if $\xi_0, \ldots, \xi_{r-1}, \eta_0, \ldots, \eta_{s-1}$ are distinct ordinal numbers $< \lambda$, then*

$$e_{\xi_0} \cdots e_{\xi_{r-1}}(1 - e_{\eta_0}) \cdots (1 - e_{\eta_{s-1}}) \neq 0.$$

(b) *R satisfies the countable chain condition; that is, if $\{x_i \mid i \in I\}$ is a set of nonzero elements in R such that $x_i x_j = 0$ whenever $i \neq j$, then I is countable. Then*

$$\text{h. dim}\, J \geqslant n.$$

It is well known that the conditions (a) and (b) are satisfied when $R = F_n$, and $J = M = [\{e_\xi \mid \xi < \lambda\}]$, where $\{e_\xi \mid \xi < \lambda\}$ is the set of free generators of F_n (see [7], pp. 43 and 72). Therefore, we obtain the following consequence of 5.1.

COROLLARY 5.2. *Let F_α be the free Boolean ring on $\aleph_\alpha$ generators. If $\alpha = n < \omega$, then*

$$\text{gl. dim}\, F_\alpha = n + 1.$$

If α is infinite, then

$$\text{gl. dim}\, F_\alpha = \infty.$$

Proof of Theorem 5.1

Let

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \to P_1 \xrightarrow{\epsilon} J \to 0$$

be the projective resolution of J that was constructed in Section 4 (using the generating set $\{e_\xi \mid \xi < \lambda\}$). Denote $Q_n = \operatorname{Im} d_n$. If h. dim $J < n$, then Q_n is projective (see [6], p. 201). We will suppose that this is the case and derive a contradiction.

For $\delta \in \Delta_n$, define x_δ as in Section 4. Denote $y_\delta = d_n(x_\delta)$. Since $\{x_\delta \mid \delta \in \Delta_n\}$ generates P_n, it follows that $[\{y_\delta \mid \delta \in \Delta_n\}] = Q_n$. Therefore, by Theorem 3.1, there is a function $f \in R^{\Delta_n}$ such that

$$Q_n = \oplus \sum_{\delta \in \Delta_n} Rf(\delta)\, y_\delta .$$

It can be assumed without loss of generality that $f(\delta)\, N(\delta) = f(\delta)$, where $N(\delta)$ is defined as in Section 4. Indeed,

$$N(\delta)\, y_\delta = N(\delta)\, d_n(x_\delta) = d_n(N(\delta)\, x_\delta) = d_n(x_\delta) = y_\delta ,$$

by the definition of x_δ. Therefore, $f(\delta)\, y_\delta = f(\delta)\, N(\delta)\, y_\delta$. We will establish two properties of f, and then proceed to show that these are contradictory.

(1) $f(\delta) \neq 0$ for all $\delta \in \Delta_n$.

In fact, there exist $\delta_1, \ldots, \delta_m$ in Δ_n, and $a_i \in R$ $(i \leqslant m)$ such that

$$y_\delta = a_0 f(\delta)\, y_\delta + a_1 f(\delta_1)\, y_{\delta_1} + \cdots + a_m f(\delta_m)\, y_{\delta_m} ,$$

where $\delta_i \neq \delta$ for $i > 0$. Therefore, for each $i > 0$, some η_i occurs in δ_i but not in δ. Thus,

$$(1 - e_{\eta_1}) \cdots (1 - e_{\eta_m})\, y_\delta = (1 - e_{\eta_1}) \cdots (1 - e_{\eta_m})\, a_0 f(\delta)\, y_\delta .$$

If $\delta = (\xi_0, \dots, \xi_{n-1})$, then the ordinal numbers $\xi_0, \dots, \xi_{n-1}, \eta_1, \dots, \eta_m$ are distinct. Therefore, by hypothesis (a) of Theorem 5.1,

$$a_0 f(\delta)(1 - e_{\eta_1}) \cdots (1 - e_{\eta_m}) y_\delta(\xi_1, \dots, \xi_{n-1})$$

$$= (1 - e_{\eta_1}) \cdots (1 - e_{\eta_m}) y_\delta(\xi_1, \dots, \xi_{n-1})$$

$$= (1 - e_{\eta_1}) \cdots (1 - e_{\eta_m}) e_{\xi_0} \cdots e_{\xi_{n-1}} \neq 0.$$

This proves (1).

Let $\{\xi_{ij} \mid i < n, j < 2\}$ be a set of $2n$ distinct ordinal numbers less than λ, where $n \geqslant 1$. For $\mu \in 2^n$, define $\delta(\mu) \in \Delta_n$ to be the rearrangement of the sequence $(\xi_{0\mu(0)}, \dots, \xi_{n-1\mu(n-1)})$ into increasing order.

(2) $\prod_{\mu \in 2^n} f(\delta(\mu)) = 0$.

For convenience, denote $\prod_{\mu \in 2^n} f(\delta(\mu))$ by c. Then $cf(\delta(\mu)) = c$ for all $\mu \in 2^n$. Let $\nu \in 2^{n-1}$. Define

$$\delta'(\nu) \in \Delta^{n+1}$$

to be the rearrangement of the sequence

$$(\xi_{0\nu(0)}, \dots, \xi_{n-2\,\nu(n-2)}, \xi_{n-1\,0}, \xi_{n-1\,1})$$

into increasing order. Define

$$\varphi = c \sum_{\nu \in 2^{n-1}} x_{\delta'(\nu)} .$$

A direct computation yields

$$d_{n+1}\varphi = c \sum_{\mu \in 2^n} x_{\delta(\mu)} .$$

Consequently,

$$0 = d_n\, d_{n+1}\varphi = c \sum_{\mu \in 2^n} y_{\delta(\mu)} = \sum_{\mu \in 2^n} cf(\delta(\mu))\, y_{\delta(\mu)} .$$

Thus, $cf(\delta(\mu))\, y_{\delta(\mu)} = 0$, which implies that

$$c = cf(\delta(\mu)) = 0.$$

The desired contradiction between (1) and (2) is expressed in the following lemma, which will complete the proof of 5.1.

LEMMA 5.3. *Let R be a Boolean ring that satisfies the countable chain condition. Let W be a set such that $|W| \geqslant \aleph_n$, where $n \geqslant 1$. Suppose that $f : W^n \to R$ is a mapping which satisfies*

$$\prod_{\mu \in 2^n} f(\tau(0, \mu(0)), \dots, \tau(n-1, \mu(n-1))) = 0, \qquad (*)$$

for all one-to-one mappings τ of $n \times 2$ into W. Then distinct elements $w_0, \ldots, w_{n-1}$ exist in W such that

$$\tilde{f}(w_0, \ldots, w_{n-1}) = 0.$$

Proof. If $n = 1$, there is nothing to prove. Therefore, we can make the induction assumption that $n > 1$ and the lemma is true for $n - 1$. For $w \in W$, define the mapping

$$\tilde{g}_w : W^{n-1} \to R \qquad \text{by} \qquad \tilde{g}_w(w_0, \ldots, w_{n-2}) = \tilde{f}(w_0, \ldots, w_{n-2}, w_{n-1}).$$

Let $\sigma \in W^{(n-1)\times 2}$ be one-to-one. Denote by $c_w{}^\sigma$ the product

$$\prod_{\nu \in 2^{n-1}} \tilde{g}_w(\sigma(0, \nu(0)), \ldots, \sigma(n-2, \nu(n-2))).$$

If v and w are distinct elements in $W - \sigma((n-1) \times 2)$, the hypothesis $(*)$ implies $c_v{}^\sigma c_w{}^\sigma = 0$. By the countable chain condition, the set

$$S_\sigma = \{w \in W - \sigma((n-1) \times 2) \mid c_w{}^\sigma \neq 0\}$$

is countable. Let $V \subset W$ be such that $V \mid = \aleph_{n-1}$. Then thet set $T = U\{S_\sigma \mid \sigma \in V^{(n-1)\times 2}, \sigma \text{ one-to-one}\}$ satisfies $\mid T \mid < \aleph_n$. Hence, $w_{n-1} \in W - (V \cup T)$ exists. The induction hypothesis applies to V and the function $\tilde{g}_{w_{n-1}}$. Accordingly, distinct elements $w_0, \ldots, w_{n-2}$ exist in V satisfying $\tilde{g}_{w_{n-1}}(w_0, \ldots, w_{n-2}) = 0$. Hence, $\tilde{f}(w_0, \ldots, w_{n-2}, w_{n-1}) = 0$ and $w_0, \ldots, w_{n-2}, w_{n-1}$ are distinct.

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